Markov Chains

Finite-Sate, Discrete-Time Markov Chains

Contents

Concepts of Markov Process

- Markov Process
  - The future state depends only on the present state

- ?? Bernoulli process and Poisson process

- MP: Random Walk, Wiener Process, Sum Process, Random Telegraph
- Not MP: Moving Average Process

- Example: Customer arrival process!

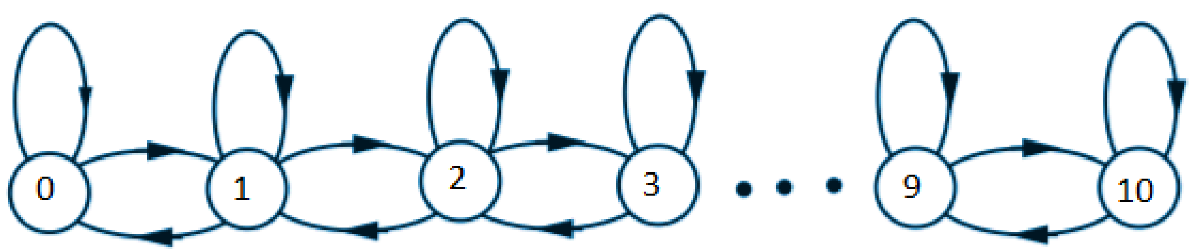
Concepts of Markov Process

- "State" X_n: Number of customers at time n.

Customer arrivals: Bernoulli(p)

geometric interarrival times

Customer service times: geometric(q)



Markov Process: Definition

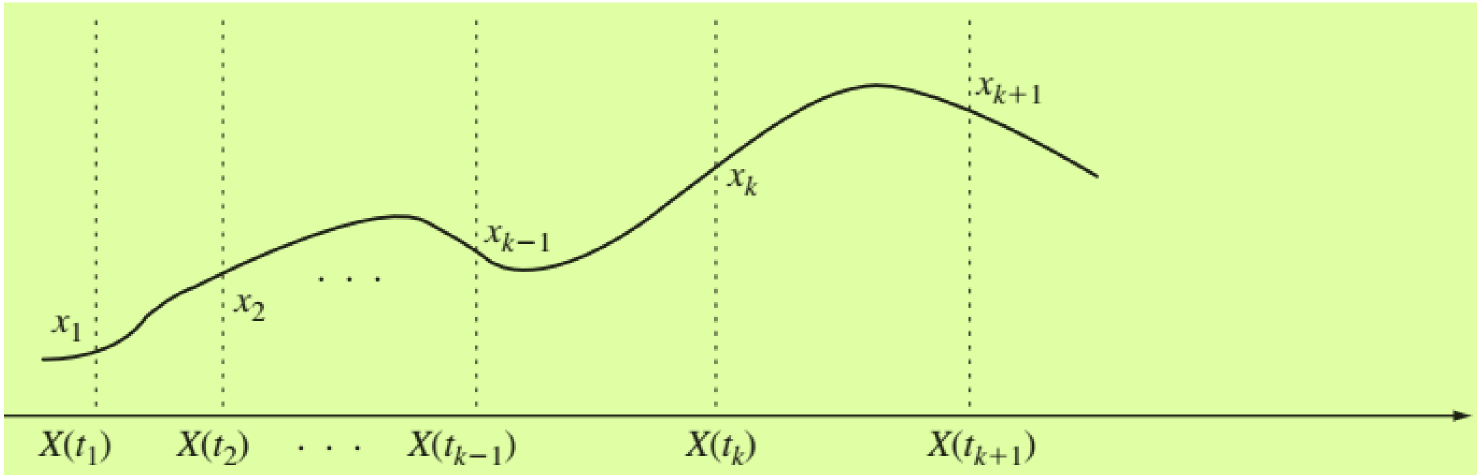
- If X(t) is discrete-valued

$$P[X(t_{k+1}) = x_{k+1} \mid X(t_k) = x_k, \ldots, X(t_1) = x_1] \\ = P[X(t_{k+1}) = x_{k+1} \mid X(t_k) = x_k]$$

- If X(t) is continuous-valued

$$P[a < X(t_{k+1}) \leq b \mid X(t_k) = x_k, \ldots, X(t_1) = x_1] \\ = P[a < X(t_{k+1}) \leq b \mid X(t_k) = x_k]$$

Markov Process: Definition



Markov property: Given $X(t_k)$, $X(t_{k+1})$ is independent of samples prior to t_k.

- If the samples of X(t) are jointly continuous

$$f_{X(t_{k+1})}(x_{k+1} \mid X(t_k) = x_k, \ldots, X(t_1) = x_1) = f_{X(t_{k+1})}(x_{k+1} \mid X(t_k) = x_k).$$

- State: pmf's and pdf's in a MP that are conditioned on several time instants always reduce to a pmf/pdf that is conditioned only on the most recent time instant.

Markov Chains

- An integer-valued Markov Process

If $X(t)$ is a Markov chain, then the joint pmf for three arbitrary time instants is

$$\begin{aligned} P[X(t_3) = x_3, X(t_2) = x_2, X(t_1) = x_1] \\ &= P[X(t_3) = x_3 | X(t_2) = x_2, X(t_1) = x_1] P[X(t_2) = x_2, X(t_1) = x_1] \\ &= P[X(t_3) = x_3 | X(t_2) = x_2] P[X(t_2) = x_2, X(t_1) = x_1] \\ &= P[X(t_3) = x_3 | X(t_2) = x_2] P[X(t_2) = x_2 | X(t_1) = x_1] P[X(t_1) = x_1], \end{aligned}$$

In General,

$$\begin{aligned} P[X(t_{k+1}) = x_{k+1}, X(t_k) = x_k, \ldots, X(t_1) = x_1] \\ &= P[X(t_{k+1}) = x_{k+1} | X(t_k) = x_k] \end{aligned}$$

$$\begin{aligned} P[X(t_k) = x_k | X(t_{k-1}) = x_{k-1}] \ldots P[X(t_1) = x_1] \\ &= \left\{ \prod_{j=1}^{k} P[X(t_{j+1}) = x_{j+1} | X(t_j) = x_j] \right\} P[X(t_1) = x_1] \end{aligned}$$

Discrete-Time Markov Chains

Let X_n be a discrete-time integer-valued Markov chain that starts at $n = 0$ with pmf

$$p_j(0) \triangleq P[X_0 = j] \qquad j = 0, 1, 2, \ldots.$$

- Assume
  - X_n is finite state
  - X_n takes on values on countable set of integers
- The joint pmf for the first n+1 values of the process

$$\begin{aligned} P[X_n = i_n, \ldots, X_0 = i_0] \\ &= P[X_n = i_n | X_{n-1} = i_{n-1}] \ldots P[X_1 = i_1 | X_0 = i_0] P[X_0 = i_0]. \end{aligned}$$

Discrete-Time Markov Chains

- One-step transition probabilities

$$P[X_{n+1} = j \mid X_n = i] = p_{ij} \qquad \text{for all } n.$$

Homogeneous

- So, the joint pmf for the first n+1 values of the process

$$P[X_n = i_n, \ldots, X_0 = i_0] = p_{i_{n-1},i_n} \ldots p_{i_0,i_1} p_{i_0}(0).$$

Thus X_n is completely specified by the *initial pmf* $p_i(0)$ and the *matrix of one-step transition probabilities* P:

Transition probability matrix

$$P = \begin{bmatrix} p_{00} & p_{01} & p_{02} & \cdots \\ p_{10} & p_{11} & p_{12} & \cdots \\ . & . & . & \\ p_{i0} & p_{i1} & \cdots & \\ . & . & \cdots & \end{bmatrix}.$$

$$1 = \sum_j P[X_{n+1} = j \mid X_n = i] = \sum_j p_{ij}.$$

Markov Chains: n-Step Transition Probabilities

$$p_{ij}(n) = P[X_{n+k} = j \mid X_k = i] \qquad n \geq 0, i, j \geq 0.$$

- Two-step transition probabilities

$$\begin{aligned}
P[X_2 = j, X_1 = k \mid X_0 = i] &= \frac{P[X_2 = j, X_1 = k, X_0 = i]}{P[X_0 = i]} \\
&= \frac{P[X_2 = j \mid X_1 = k]P[X_1 = k \mid X_0 = i]P[X_0 = i]}{P[X_0 = i]} \\
&= P[X_2 = j \mid X_1 = k]P[X_1 = k \mid X_0 = i] \\
&= p_{ik}(1)p_{kj}(1).
\end{aligned}$$

Markov Chains: n-Step Transition Probabilities

$$p_{ij}(2) = \sum_k p_{ik}(1)p_{kj}(1) \qquad \text{for all } i, j.$$

$$P(2) = P(1)P(1) = P^2.$$

Chapman–Kolmogorov equations:

$$p_{ij}(m+n) = \sum_k p_{ik}(m)p_{kj}(n) \quad \text{for all } n, m \geq 0 \text{ all } i, j.$$

$$P(n+m) = P(n)P(m).$$

$$P(n) = P^n.$$

Markov Chains: State Probabilities

- State probabilities at time *n*

$$p_j(n) = \sum_i P[X_n = j \mid X_{n-1} = i]P[X_{n-1} = i]$$
$$= \sum_i p_{ij}p_i(n-1).$$

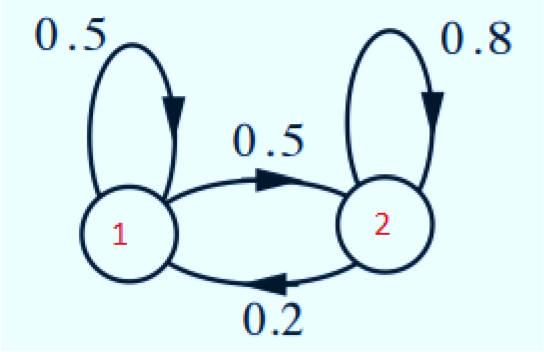
$$\mathbf{p}(n) = \mathbf{p}(n-1)P.$$

Similarly, $p_j(n)$ is related to $\mathbf{p}(0)$ by

$$p_j(n) = \sum_i P[X_n = j \mid X_0 = i]P[X_0 = i]$$
$$= \sum_i p_{ij}(n)p_i(0),$$

$$\mathbf{p}(n) = \mathbf{p}(0)P(n) = \mathbf{p}(0)P^n \qquad n = 1, 2, \ldots.$$

Markov Chains: State Probabilities



	$n=0$	$n=1$	$n=2$	$n=100$	$n=101$
$r_{11}(n)$	1	0.5	.35	2/7	?
$r_{12}(n)$	0	0.5	.65	5/7	?
$r_{21}(n)$	0	0.2			
$r_{22}(n)$	1	0.8			

Markov Chains: Steady State Probabilities

- As n approaches infinite $p_{ij}(n) \rightarrow \pi_j \quad \text{for all } i.$ $P^n \rightarrow \mathbf{1}\boldsymbol{\pi}$

"Equilibrium" or "Steady state"

$$p_j(n) = \sum_i p_{ij}(n) p_i(0) \rightarrow \sum_i \pi_j p_i(0) = \pi_j.$$

$$\pi_j = \sum_i p_{ij} \pi_i,$$

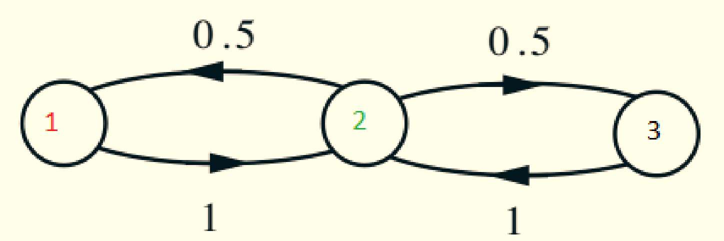
$$\boldsymbol{\pi} = \boldsymbol{\pi} P.$$

Underdetermined sys

$$\mathbf{p}(n) = \boldsymbol{\pi} P^n = \boldsymbol{\pi} \qquad \text{for all } n.$$

Convergence of the Chain

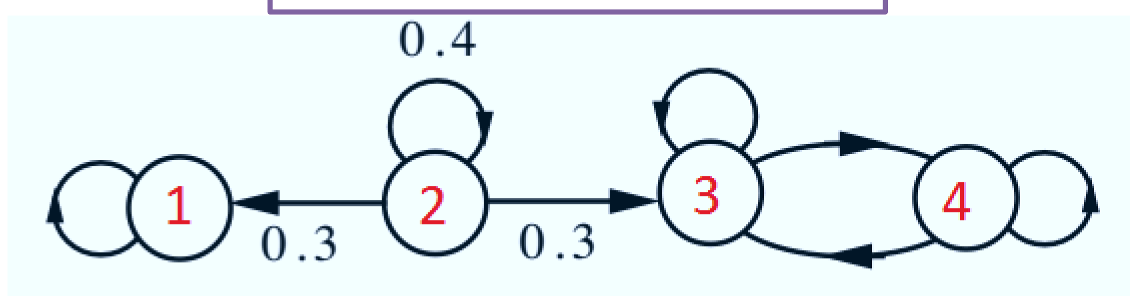
$P_{ij}(n)$ Converge?



n: even : P22=1

n: odd: P22=0

Does the Initial State matter?



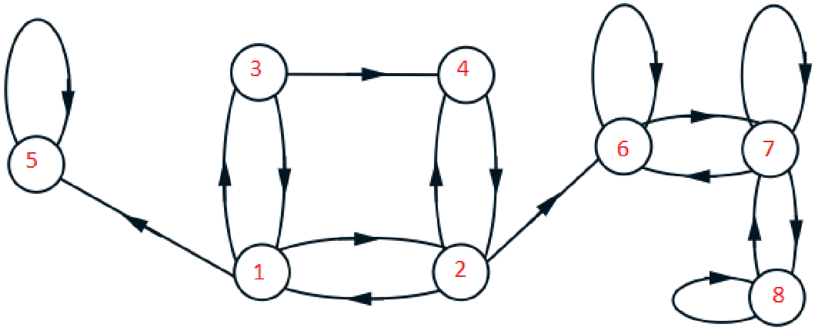
P11(n)=1 for all n

P31(n)=0 for all n

Classes of States

- The states of a Markov chain consist of one or more disjoint communication classes.

- A Markov chain that consists of a single class is said to be irreducible



State i is **recurrent** if:
starting from i,
and from wherever you can go,
there is a way of returning to i

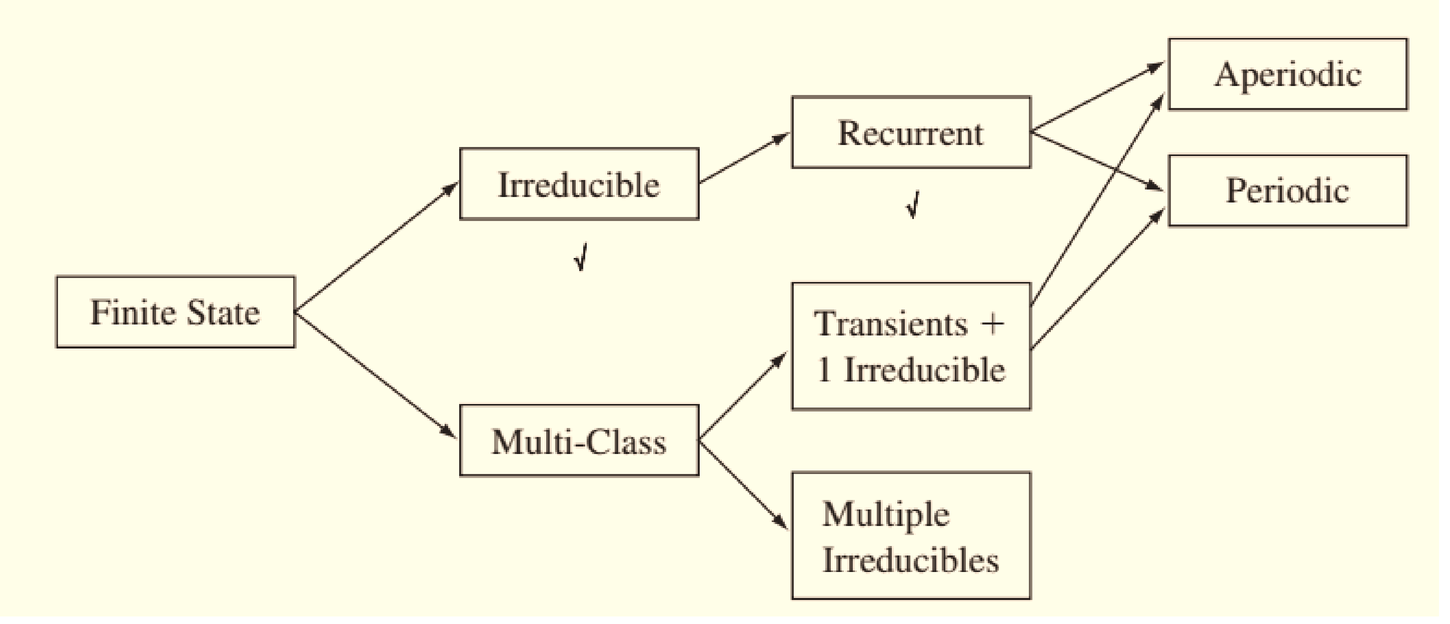
If not recurrent, called **transient**

$$f_i = P[\text{ever returning to state } i] = 1.$$

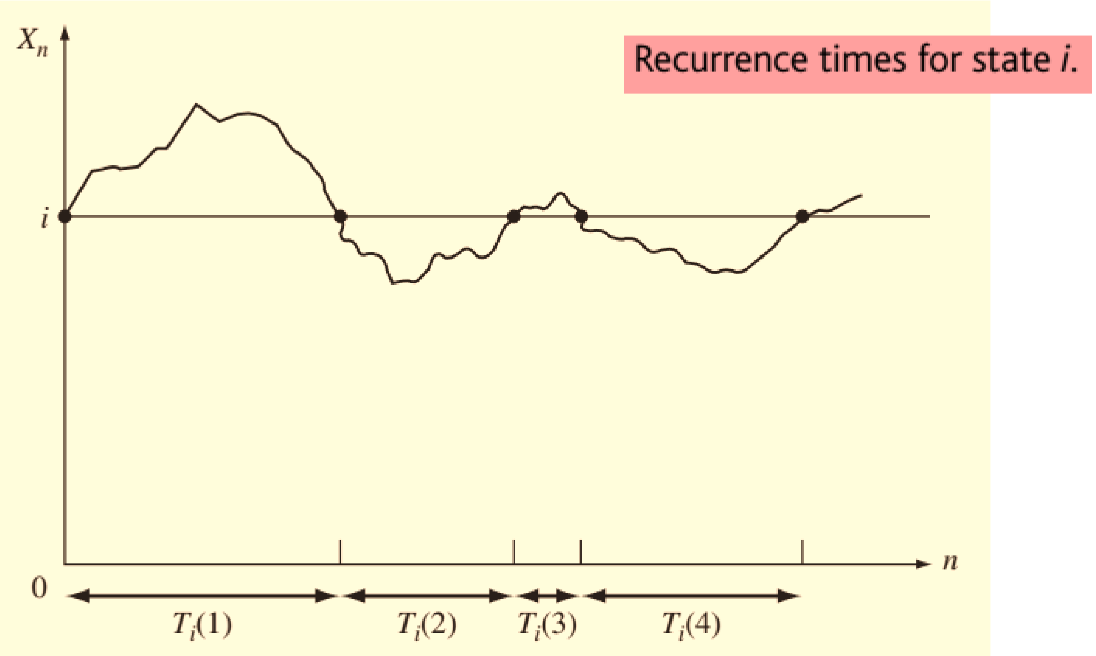
$$\sum_{n=1}^{\infty} p_{ii}(n) = \infty.$$

$$f_i < 1. \quad \sum_{n=1}^{\infty} p_{ii}(n) < \infty.$$

Finite State Markov Chain: structure



Limiting Probabilities



$$\text{proportion of time in state } i = \frac{k}{T_i(1) + T_i(2) + \cdots + T_i(k)}.$$

$$\text{proportion of time in state } i \rightarrow \frac{1}{E[T_i]} = \pi_i,$$

If $E[T_i] < \infty$, then we say that state i is **positive recurrent**.

$\pi_i > 0$ if state i is positive recurrent.

If $E[T_i] = \infty$, then we say that state i is **null recurrent**.

$\pi_i = 0$ if state i is null recurrent.